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A theory for the laminar wake of a two-dimensional body in a boundary layer

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A new theory is developed for the wake far downstream of a cylindrical body of height h, placed with its generators perpendicular to the flow on a surface above which there is a boundary layer of thickness δ. If the streamwise (x) velocity in the wake is $(U+u)$, then assuming (h/δ) is small enough that the velocity profile in the boundary layer may be regarded as $U = \alpha y$, and assuming $|u| \ll U$, linear differential equations governing u are derived. It is found that a constant along the wake is

$$I = \frac{3}{2}\int_0^\infty yUu\,dy.$$

This result can be used to find an order of magnitude estimate for u, because I is related to the forces on the body producing the wake by the approximate formula

$$I \simeq -C_1/\rho,$$

where C_1 is that component of the couple on the body produced by pressure and viscous stresses in the x direction. For the particular case of a small hump on the boundary of height h and length b, such that $h \ll b$, the above relation is shown to be exact. The perturbation velocity in the wake is found to have a similarity solution

$$u = [I/(x\nu)]f(y^3/[x\nu/\alpha]),$$

the physical implications of which are discussed in detail. The relevance of the theory to the problem of transition behind a trip wire is also mentioned.

1. Introduction

One of the most illuminating experiments in fluid dynamics is the demonstration (due to Prandtl) of the effect of transition on separation by placing a trip wire in the boundary layer of a sphere (Goldstein 1938, p. 72). Although there have been experiments to find out the details of how the wire creates the transition, there have been no attempts to calculate the effect of the wire on the velocity profile in the boundary layer and thence on its stability. So this is one good reason for studying the laminar wake downstream of a two-dimensional body placed in a boundary layer. However, the main reason is that there are many practical problems for which an underestimating of these kind of wakes is important. Although, in general, the wakes are turbulent, e.g. behind buildings in the atmospheric boundary layer, or roughness elements on aircraft wings, a

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fundamental understanding of the equivalent laminar flow problem is always necessary before embarking on a phenomenological analysis of a turbulent flow, this being our ultimate intention.

Some aspects of the problem of a laminar wake in a boundary layer have been examined implicitly by Goldstein (1938), in that he analyzed the flow in a boundary layer when the distribution of velocity is given as a power series in y at a plane $x = x_0$, say, However, this does not really help because essentially a wake far downstream of a body emanates from a delta function at the body, and such a delta function cannot easily be represented in terms of Goldstein's power series. Thus Goldstein's method has to be rejected. On the other hand, the problem has been explicitly examined by Gertsenshtein (1966) who uses a Galerkin method to calculate the flow over a semi-circular cylinder placed in a boundary layer. However his linearized equations are not correct near the body, unless the body's Reynolds number, R_h, is very small. Since he calculates wake flows when R_h is of order 10, his results must be regarded as suspect.

The method adopted here is first to find the linear differential equations governing a perturbation on the boundary-layer flow near the surface $y = 0$ (§ 2). From these equations it follows that there is an integral, I, of the perturbation velocity, u, which is constant along the wake. In § 3 we examine the relation between I and the forces on the body. In § 4 we look for a similarity solution to the equations of § 2, anticipating that there must be a similar solution for these wakes far enough downstream. As a check on this solution we obtain in § 5 a uniformly valid asymptotic solution for flow over a small hump of height h_0, length b, such that $h_0 \ll b$. The analysis shows that, as $x \to \infty$, the solution tends to the similarlity solution and also enables us to compare I with the calculated forces on the body. In § 6 we discuss the conditions under which the analysis for an arbitrary body is valid and in § 7 mention some preliminary conclusions about the stability of the perturbed flow in the wake.

The first detailed experimental investigations into the flow behind boundary-layer trip wires were measurements of velocity profiles behind wires of varying sizes by Liepmann & Fila (1947). These experiments were repeated and extended by Tani & Sato (1956), and recently by Hall (1968). Tani & Sato also measured the fluctuating velocities in the wakes as they became unstable, These experiments, and some interesting earlier ones of Fage (1943), were all concentrated directly on the transition problem. Their limitations in providing a check on the theory are mentioned in § 7.

2. Derivation of the governing equations

We consider the flow far downstream of a two-dimensional body of height h situated at the bottom of a boundary layer, the thickness of which is δ (see figure 1). (The analysis also applies if the body is immersed in a Couette flow or a plane Poiseuille flow, but we shall not refer to these cases hereafter.) Thus our first assumption is that the body is sufficiently small, that

$$h \ll \delta. \tag{2.1}$$

Our second assumption is to assume the truth of the hypothesis (H) that, if (2.1) is satisfied, far enough downstream the boundary-layer velocity profile reverts to its upstream form. We are not able to state sufficient conditions for

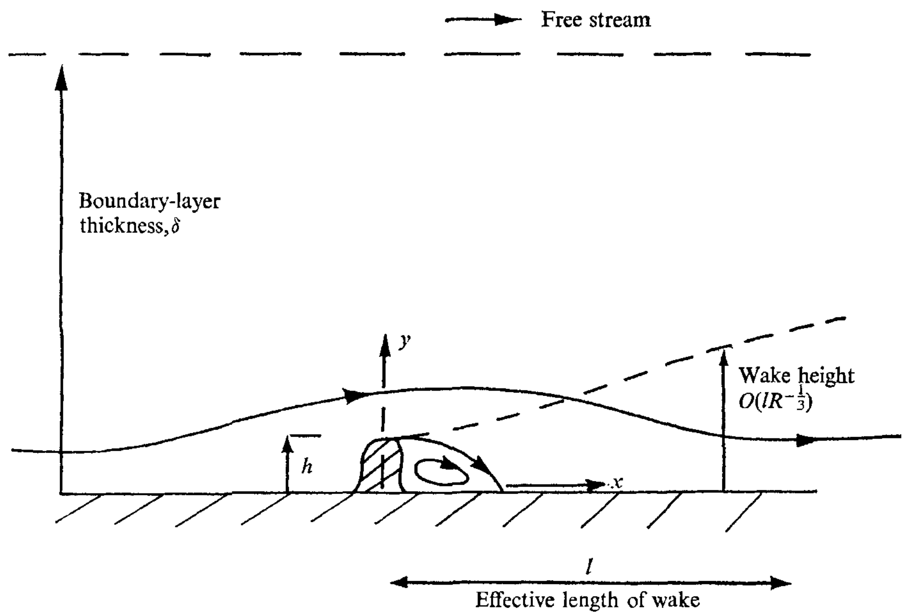

FIGURE 1. Flow over a two-dimensional body showing the wake and boundary-layer regions, and a typical streamline.

the validity of this hypothesis, but we can say that the extensive velocity profile measurements of Hall (1968) do tend to support it provided (2.1) is satisfied. One further assumption is that the Reynolds number of the wake, based on the velocity gradient at $y = 0$, $\alpha(x)$, and the distance l in which the wake decays, is large, i.e.

$$\alpha l^2/\nu = R \gg 1, \tag{2.2}$$

where ν is the kinematic viscosity. (We postpone discussion of the order of magnitude of R_h $(= \alpha h^2/\nu)$ until later.)

With u_i and p being the velocity and pressure in the wake, and $(U(x,y), V(x,y), 0)$ and P being the velocity components and pressure of the undisturbed boundary-layer flow (or pipe flow), we now define the perturbation velocity component and perturbation pressure u, v and $\tilde{p}$ as

$$u_1 = u + U, \quad u_2 = v + V, \quad p = P + \tilde{p}. \tag{2.3}$$

Since we assume the wake only occupies the bottom part of the boundary layer, we can express U as

$$U = \alpha(x)\,y + \beta(x)\,y^2 + \ldots, \tag{2.4}$$

with the higher-order terms negligible in the region of interest. It follows that the appropriate scaling for a perturbation on U is that used by Goldstein (1930) namely:

$$\left.\begin{aligned} x^* &= x/l, \quad y^* = R^{\frac{1}{3}}y/l, \\ u^* &= [\epsilon\alpha l R^{-\frac{1}{3}}]^{-1}\,u, \quad v^* = [\epsilon\alpha l R^{-\frac{2}{3}}]^{-1}\,v, \\ p^* &= \frac{1}{\epsilon\rho}(\alpha l R^{-\frac{1}{3}})^{-2}\,\tilde{p}, \end{aligned}\right\} \tag{2.5}$$

where ρ is the density and R is defined by and satisfies (2.2). From our hypothesis (H), when $x \sim l$, $|u| \ll U$, and therefore, if we define u' to be $O(1)$,

$$\epsilon \ll 1. \tag{2.6}$$

Since, by definition, U, V and P satisfy the Navier-Stokes equations, on substituting the expressions of (2.3) and (2.4) into these equations we are only left with terms of $O(\epsilon)$ and $O(\epsilon^2)$. If we ignore the latter, and also terms of $O(R^{-\frac{2}{3}})$ compared with those of $O(1)$, we obtain the following equations:

$$\begin{aligned} y^*(1+(\beta/\alpha)\,lR^{-\frac{1}{3}}\,y^*+\ldots)\,\partial u^*/\partial x + (1+2(\beta/\alpha)\,lR^{-\frac{1}{3}}\,y^*+\ldots)\,v^* \\ +\,\partial p^*/\partial x^* - \partial^2 u^*/\partial y^{*2} = \left(\frac{l}{\alpha}\frac{\partial\alpha}{\partial x}\right)\left\{u^*y^* - \tfrac{1}{2}y^{*2}\frac{\partial u^*}{\partial y^*}+\ldots\right\}, \end{aligned} \tag{2.7}$$

$$0 = -\,\partial p^*/\partial y^*, \tag{2.8}$$

$$\partial u^*/\partial x^* + \partial v^*/\partial y^* = 0. \tag{2.9}$$

The term on the right-hand side of (2.7) is essentially l/L, where L is the distance in which the boundary flow changes. Before obtaining the solution for $u^*(x^*, y^*)$ it is not possible to assess the size of this term, which it is plausible to assume is negligible if $h \ll \delta$. However, we have to proceed heuristically and assume that

$$l \ll L \tag{2.10}$$

and therefore the right-hand side of (2.7) is zero. To justify this we show, *a posteriori*, in §6 that, if (2.1) is well satisfied, then (2.10) must also be satisfied. The boundary conditions on (2.7) to (2.9) are that

$$u^* = v^* = 0 \quad \text{at} \quad y^* = 0 \quad \text{and as} \quad x^* \to \infty, \tag{2.11}$$

and, by the definition of a wake, $u^* \to 0$ as $y^* \to \infty$.

The solution of (2.7) clearly depends on $\partial p^*/\partial x^*$ about which we have no information from (2.7) alone. In addition the fact that

$$\int_0^\infty u^*\,dy^*$$

is not constant along the wake (as shown later) means that as $y^* \to \infty$, $v^* \to v^*_\infty(x)$, a function as yet unknown. Therefore to understand about p^* and the efflux or inflow v^*_∞ we must analyze the disturbed flow outside the wake, which we shall call the external flow and denote by a double asterisk. The relevant scaling length is now l in the x and y directions and the velocity scaling is chosen so that

$$v^{**} = v^*_\infty \quad \text{as} \quad y^{**} \to 0.$$

Then

$$\left.\begin{aligned} x^{**} &= x/l, \quad y^{**} = y/l, \\ u^{**} &= (\epsilon\alpha l R^{-\frac{2}{3}})^{-1}\,u, \quad v^{**} = (\epsilon\alpha l R^{-\frac{2}{3}})^{-1}\,v, \\ p^{**} &= (1/\epsilon\rho)\,(\alpha l R^{-\frac{1}{3}})^{-2}\,\tilde{p}. \end{aligned}\right\} \tag{2.12}$$

Then to $O(\epsilon)$ the equations for u^{**}, v^{**} are

$$y^{**}(1+(\beta/\alpha)\,ly^{**}+\ldots)\,\partial u^{**}/\partial x^{**}+(1+2(\beta/\alpha)\,ly^{**})\,v^{**} = -\,\partial p^{**}/\partial x^{**}, \quad (2.13)$$

$$y^{**}(1+(\beta/\alpha)\,ly^{**}+\ldots)\,\partial v^{**}/\partial x^{**} = -\,\partial p^{**}/\partial y^{**}, \quad (2.14)$$

$$\partial u^{**}/\partial x^{**}+\partial v^{**}/\partial y^{**} = 0, \quad (2.15)$$

where terms of $O(R^{-1})$ and $O[(l/\alpha)\,d\alpha/dx]$ have been ignored. This external flow, is, therefore, an inviscid perturbation on the boundary-layer flow. The boundary conditions for (2.13) to (2.15) follow from the fact that u, v must be continuous where the external flow meets the wake, and from the hypothesis (H): as

$$\left.\begin{aligned} y^{**} \to 0, \quad v^{**} &= v^*_\infty(x), \\ (x^{**2}+y^{**2}) \to \infty, \quad u^{**}, v^{**} &\to 0, \end{aligned}\right\} \quad (2.16)$$

these conditions being adequate to determine u^{**}, v^{**} everywhere. Note that $u^{**}(y^{**} = 0)$ is not specified by matching with the inner flow because $u^* \to 0$ as $y^* \to \infty$.

The particular scaling of (2.12) is only reasonable if the external flow extends in the y direction a distance small compared with δ, i.e. $l \ll \delta$. This will only be true for very small protuberances. If $\delta = O(l)$, then the length scale becomes δ and the velocity scale U_1, the free-stream velocity. In that case the flow outside the boundary layer is affected. This procedure of dividing up the flow in the boundary layer into an inner viscous region, an intermediate inviscid region, and possibly another inviscid region outside the boundary layer has been used by Stewartson (1969) to examine the trailing edge of a flat plate. He calls this a 'triple deck structure'. In the flows we consider here, two 'decks' only are needed, namely those governed by (2.7) to (2.9) and (2.13) to (2.15).

Hereafter we concentrate on the simplest wakes where the boundary-layer profile $U(y)$ is such that either $\beta = 0$, i.e. a boundary layer with no external pressure gradient ($\partial p/\partial x = 0$), or where

$$(\beta/\alpha)\,l \ll 1.$$

Then in (2.7), (2.13) and (2.14) $U(y)$ becomes a simple shear flow, $U = \alpha y$. Since p is continuous at the boundary between the external flow region and the wake, (2.13) shows that

$$\partial p^*/\partial x^* = (\partial p^{**}/\partial x^{**})\,(y^{**} \to 0) = -\,v^*_\infty.$$

Thence the equations governing u^*, v^* in the viscous wake become

$$y^*\,\partial u^*/\partial x^*+v^*-v^*_\infty = \partial^2 u^*/\partial y^{*2}, \quad (2.17)$$

$$\partial u^*/\partial x^*+\partial v^*/\partial y^* = 0, \quad (2.18)$$

with boundary conditions

$$\left.\begin{aligned} u^* = v^* = 0 \quad &\text{at} \quad y^* = 0, \\ u^* \to 0 \quad &\text{as} \quad y^* \to \infty. \end{aligned}\right\} \quad (2.19)$$

Equation (2.19) gives no indication as to the boundary conditions on u, v at any station of x, and consequently the size of u and v are not determined. This problem is resolved, with the technique Glauert (1956) used for a wall jet, by integrating (2.17) with respect to y^* from y^* to ∞, and then again with respect to y^* from 0 to ∞. We find

$$\frac{\partial}{\partial x^*}\int_0^\infty\left(\int_{y^*}^\infty y'u(y')\,dy'\right)dy^*+\int_0^\infty\left(\int_{y^*}^\infty (v^*-v^*_\infty)\,dy'\right)dy^* = 0.$$

Using (2.18), and integrating by parts, it follows that

$$\left.\begin{aligned}\int_0^\infty \tfrac{3}{2}y^{*2}u^*\,dy^* &= I^*,\\ \text{or, in dimensional variables,}\quad \int_0^\infty \tfrac{3}{2}\alpha\, y^2 u\, dy &= I,\end{aligned}\right\}\tag{2.20}$$

where I is a constant. The physical significance of I is examined in the next section.

3. The relation between the forces exerted on a body and the flow in its wake

Consider Prandtl's well-known result (Prandtl & Tietjens 1934) that the velocity in the wake behind a body in uniform flow is related to its drag, D, by the formula

$$D = -\rho\int_{-\infty}^{\infty} Uu\,dy. \tag{3.1}$$

This suggests that there may be a relation between the constant, I, for the wake behind a body in a boundary layer and the forces on the body. We first note that

$$I = \frac{3}{2}\int_0^\infty yUu\,dy,$$

is a quantity proportional to the deficit in angular momentum in the wake produced by velocities in the x direction. This suggests that

$$I \propto C_1, \tag{3.2}$$

C_1 being the contribution to the couple on the body produced by stresses in the x direction.

Consider the following integral of the equation governing the x momentum of the fluid

$$\iint_A y\left\{-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xx}}{\partial x}-\rho\left(u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}\right)\right\}dx\,dy = 0 \tag{3.3}$$

over the area, A, that is outside the body and within the box shown in figure 2, the faces of which are at $x = X_1, X_2$; $y = 0, Y$. X_1, X_2 are assumed to be $O(l)$ and the line $y = Y$ is assumed to be just outside the viscous wake, so that $Y/l \ll 1$ and $YR^{\frac{1}{3}}/l \gg 1$. τ_{xy} and τ_{xx} are the viscous shear stresses given by

$$\tau_{xy} = \eta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\quad \tau_{xx} = 2\eta\frac{\partial u}{\partial x}. \tag{3.4}$$

Now we use the suffixes $+$, $-$ to denote the values of p, τ_{xy}, τ_{xx} on the downstream and upstream faces at a given value of h, $y = h(x)$ being the profile of the body, where $h \leqslant h_0$ and $|x| \leqslant b$. We find that if the integral in (3.3) is re-arranged

$$C_1/\rho = \int_0^Y y\left[\tfrac{1}{2}(u_{1_1}^2 - u_{1_2}^2) - (p_2 - p_1)/\rho + (\tau_{xx_2} - \tau_{xx_1})/\rho\right] dy + (1/\rho)\int_{x_1}^{x_2} Y\tau_{xy_4}\, dx - \iint_A [yu_2\, \partial u_1/\partial y + \tau_{xy}/\rho]\, dy\, dx, \quad (3.5)$$

where, by definition,

$$C_1 = \int_0^{h_0} y[(p_- - p_+) + (\tau_{xx_+} - \tau_{xx_-})]\, dy + \int_{-b}^{b} h(x)\tau_{xy}\,(y = h(x))\, dx, \quad (3.6)$$

the lower suffix referring to the face of the box or the body on which the integral is applied.

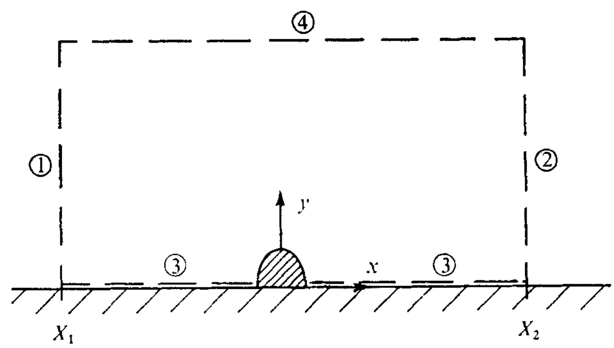

FIGURE 2. Control surface for calculating the couple on the body at $x = 0$. The numbers refer to those faces of the surface for which the integrals in § 3 apply.

In deriving (3.6) we have assumed $u_1 = u_2 = 0$ on the body and the ground. Now, using (3.4) and the assumptions about the far wake of § 2, it follows that the shear stress terms in (3.5) are negligible. In order to express (3.5) solely in terms of the velocity on the sides of the box, we have to make an Oseen type of approximation that

$$u_2\, \partial u_1/\partial y \simeq u_2\, \partial U/\partial y. \quad (3.7)$$

This approximation is likely to be valid for a streamline body with no separation. In § 5 we show that it is exact to first order. But for a blunt body this approximation is clearly invalid in the 'bubble' behind the body. However, since the velocity in the bubble is small, the contribution to (3.5) by this region may not be very important.†

Since $\partial p/\partial y = 0$ in the wake, and since on $y = Y$, from (2.13),

$$\partial p/\partial x = -\rho v\, \partial U/\partial y,$$

it follows that

$$\int_0^Y y(p_2 - p_1)/\rho\, dy = -\tfrac{1}{2}Y^2\alpha \int_{X_1}^{X_2} u_{2_4}\, dx,$$

$$= -\tfrac{1}{2}Y^2\alpha \int_0^Y (u_{1_1} - u_{1_2})\, dy. \quad (3.8)$$

† If R_h is large enough, Batchelor's (1956) theory shows that the vorticity in the bubble is constant. In that case the contribution by the bubble to (3.5) is exactly zero.

Since $u(x = X_1)$ is negligible compared with $u(x = X_2)$, being $O(\epsilon R^{-\frac{2}{3}})$ compared with $O(\epsilon R^{-\frac{1}{3}})$, (3.5) gives

$$C_1/\rho \simeq \int_0^Y \{-\alpha y^2 u(x = X_2) - \tfrac{1}{2}\alpha y^2 u(x = X_2)\}\, dy$$
$$-\tfrac{1}{2}Y^2\alpha \int_0^Y u(x = X_2)\, dy + \tfrac{1}{2}\alpha Y^2 \int_0^Y u(x = X_2)\, dy,$$
$$C_1/\rho \simeq -\tfrac{3}{2}\alpha \int_0^Y y^2 u(x = X_2)\, dy. \tag{3.9}$$

where u is the perturbation velocity. Since $u \to 0$ as $y \to \infty$, we can let $Y \to \infty$ and since (2.20) shows that the integral on the right of (3.9) is constant whatever the value of X_2, (3.9) becomes

$$C_1 \simeq -\rho I, \tag{3.10}$$

where
$$I = \int_0^\infty \tfrac{3}{2}\alpha y^2 u\, dy.$$

Now C_1 is the contribution to the total couple on the body caused by pressure and viscous forces acting in the x direction. In tensor notation, if dS_j is the vector normal to an element of area dS, and δ_{ij} is the Kronecker delta,

$$C_1 = \int_S x_2(\tau_{1j} - p\delta_{1j})\, dS_j,$$

S being the total area. Clearly the total couple

$$C = C_1 + C_2,$$

where
$$C_2 = \int_S x_1(\tau_{2j} - p\delta_{2j})\, dS_j.$$

Only if the body is a plate normal to the flow is $C_2 = 0$.

Note that our result (3.10) is only exact if (3.7) is exact, which is true in the rather degenerate example discussed in § 5. How true it is in general can only be tested by numerical computation of the full non-linear equations.

4. Solutions for the wake and external flow

We now obtain the simplest and most plausible solution for the 'inner viscous wake' equations (2.17) and (2.18), subject to the boundary conditions (2.19) and (2.20).

Differentiating (2.17) with respect to y^*, we obtain

$$y^*\, \partial^2 u^*/\partial x^*\, \partial y^* = \partial^3 u^*/\partial y^{*3}. \tag{4.1}$$

Now *assume* that far downstream the wake appears to emanate from the line $x = y = 0$ and that the velocity profile is similar at all values of x. Then

$$\left.\begin{aligned} u^* &= x^{*k} f(\eta), \\ \eta &= y^{*3}/x^* = y^3/(x\nu/\alpha). \end{aligned}\right\} \tag{4.2}$$

where

Then (4.1) becomes

$$x^{*(k-1)}\{3\eta(1-k)f' + 3\eta^2 f'' + 6f' + 54\eta f'' + 27\eta^2 f'''\} = 0. \tag{4.3}$$

k is found by using (2.20), whence

$$\begin{aligned}\frac{3}{2}\int_0^\infty y^{*2}x^{*k}f(\eta)\,dy^* &= \tfrac{1}{2}x^{*(k+1)}\int_0^\infty f(\eta)\,d\eta \\ &= I^*.\end{aligned} \tag{4.4}$$

Since I^* is a constant, it follows that

$$k = -1,$$

and therefore (4.3) leads to the equation for $f(\eta)$:

$$\eta^2 f''' + [\tfrac{1}{9}\eta^2 + 2\eta]f'' + 2(1+\eta)\tfrac{1}{9}f' = 0. \tag{4.5}$$

The boundary conditions on f are

$$\left.\begin{aligned} f(0) = f(\infty) &= 0, \\ \text{and} \qquad \int_0^\infty f(\eta)\,d\eta &= 2I^*,\end{aligned}\right\} \tag{4.6}$$

which are not sufficient to determine the solution to the third-order differential equation, (4.5). However, if we integrate (4.5) from 0 to η, it follows that

$$\eta^2 f'' + (\tfrac{1}{9}\eta^2)f' + (\tfrac{2}{9})f = [\eta^2 f'' + (\tfrac{1}{9}\eta^2)f' + (\tfrac{2}{9})f]_{(\eta=0)}.$$

Now $\eta^2 f''$ and $\eta^2 f'$ must both be zero as $\eta \to 0$, if the pressure gradient and shear stress are not to be singular at $y = 0$. Therefore we have reduced the equation to the second-order one

$$f'' + \tfrac{1}{9}f' + [2/(9\eta^2)]f = 0, \tag{4.7}$$

to which the solution is

$$f = G\eta^{\frac{1}{2}}\exp\{-(\tfrac{1}{18}\eta)\}K_{\frac{1}{6}}(\tfrac{1}{18}\eta), \tag{4.8}$$

where G is a constant and $K_{\frac{1}{6}}(\frac{1}{18}\eta)$ is a modified Bessel function. Using the result, quoted by Luke (1962), that

$$\int_0^\infty t^\mu e^{-t}K_\nu(t)\,dt = \frac{\Gamma(\mu+\nu+1)\,\Gamma(\mu-\nu+1)}{2^\mu\Gamma(\frac{3}{2}+\mu)/\Gamma(\frac{3}{2})},$$

$$G = 2I(\alpha/\nu^3)^{\frac{1}{2}}\frac{2^{\frac{1}{2}}\Gamma(2)/\Gamma(\frac{3}{2})}{18^{\frac{3}{2}}\Gamma(\frac{5}{3})\,\Gamma(\frac{4}{3})}.$$

The final solution for u is

$$\left.\begin{aligned} u^* &= (I^*/x^*)\,F(\eta), \\ \text{or} \qquad u &= (I/[\nu x])\,F(\eta), \\ \text{where} \qquad F(\eta) &= \frac{1}{2\pi\,(3\pi)^{\frac{1}{2}}}\eta^{\frac{1}{2}}\exp(-\tfrac{1}{18}\eta)\,K_{\frac{1}{6}}(\tfrac{1}{18}\eta).\end{aligned}\right\} \tag{4.9}$$

The general form of u can be seen from the asymptotic properties of u^*. We find as $\eta \to 0$,

$$u^* \sim \frac{I^*}{2\pi(3\pi)^{\frac{1}{2}}}\left(\frac{\eta^{\frac{1}{3}}(36)^{\frac{1}{6}}\,\Gamma(\frac{1}{6})}{2x^*}\right), \tag{4.10}$$

or

$$u^* \propto y^*/x^{*\frac{4}{3}},$$

whereas when $\eta \to \infty$

$$u^* \sim \frac{I^*}{x^*}\left(\frac{(\frac{1}{2}\pi)^{\frac{1}{2}}}{2\pi(3\pi)^{\frac{1}{2}}}\right)e^{-\frac{1}{9}\eta}. \tag{4.11}$$

These results agree with the values of u^* computed from (4.19) which, expressed as $u/(I/[\nu x])$, are plotted against $y/(x\nu/\alpha)^{\frac{1}{3}}$ in figure 3. As a check on the computations of $K_{\frac{1}{6}}$, $f(\eta)$ was plotted against η, and the area under the curve found to be within 2 % of the exact value of $\int_0^\infty f(\eta)\,d\eta$ calculated analytically. From figure 3

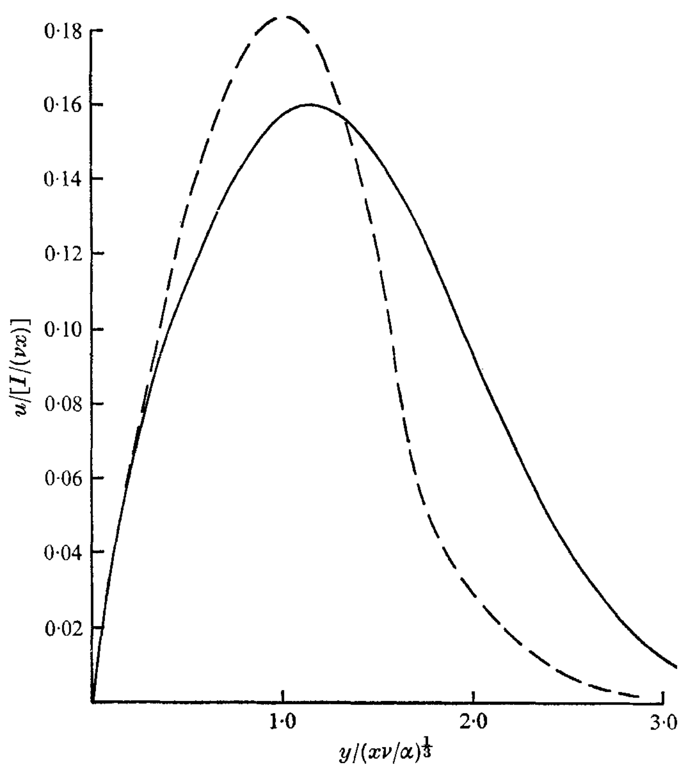

FIGURE 3. Velocity profile in a laminar wake. ——, exact solution (4.9); – – –, approximate solution (4.18).

we see that the line on which the maximum velocity deficit in the wake is the cubic $y^3 = 1{\cdot}52\,(x\nu/\alpha)$, and a line approximately marking the edge of the wake is $y^3 = 3(x\nu/\alpha)$. Note that in (4.9), given I, u is independent of the value of l. As with Prandtl's solution ours also is independent of R_h.

From (4.9) and the continuity equation v^*_∞ is found to be

$$v^*_\infty = I^* A x^{*-\frac{5}{3}}, \tag{4.12}$$

where

$$A = 8\pi^{\frac{1}{2}}\,\Gamma(\tfrac{2}{3})/[9^{\frac{1}{3}}\Gamma^3(\tfrac{1}{3})].$$

From (4.12) we can now calculate the flow in the external flow. It follows from (2.13) and (2.14) that the vorticity of this flow is given by

$$y^{**}\left(1+\frac{\beta}{\alpha}ly^{**}+\ldots\right)\frac{\partial}{\partial x^{**}}\left(\frac{\partial v^{**}}{\partial x^{**}}-\frac{\partial u^{**}}{\partial y^{**}}\right)-v^{**}\left(2\frac{\beta}{\alpha}l+\ldots\right)=0.$$

In the particular case $U-\alpha y$, since v^{**} and u^{**} far upstream are zero,

$$\partial v^{**}/\partial x^{**}-\partial u^{**}/\partial y^{**}=0 \tag{4.13}$$

throughout the external flow region. The external flow is partly produced by the wake acting as a sink, and also by the doublet effect of the body itself. Since the latter flow decreases like r^{-2} from the body, compared with $r^{-\frac{4}{3}}$ for the former, we can ignore it. Putting $u^{**}=\partial\psi/\partial y^{**}$ and $v^{**}=-\partial\psi/\partial x^{**}$ from (4.13) it follows that

$$\partial^2\psi/\partial x^{**2}+\partial^2\psi/\partial y^{**2}=0. \tag{4.14}$$

The boundary conditions on ψ are

$$\left.\begin{array}{lll}\partial\psi/\partial x^{**}=v_\infty & \text{on} & y^{**}=0, x^{**}>0;\\ \partial\psi/\partial x^{**}=0 & & y^{**}=0, x^{**}<0;\\ \text{and}\quad |\nabla\psi|\to 0 & \text{as} & (x^{**2}+y^{**2})\to\infty.\end{array}\right\} \tag{4.15a,b,c}$$

Clearly near $x=0$ these conditions are not realistic, as the wake solution is invalid there also. The solution to (4.14) subject to (4.15) is very simple, being

$$\psi=\left(\frac{I}{\nu}\right)\left(\frac{\nu}{\alpha}\right)^{\frac{1}{3}}\tfrac{3}{2}A\,r^{-\frac{2}{3}}\left(\cos\left(\tfrac{2}{3}\theta\right)+\frac{\sin\left(\frac{2}{3}\theta\right)}{3^{\frac{1}{2}}}\right), \tag{4.16}$$

where $r=(x^{**2}+y^{**2})^{\frac{1}{2}}$ and $\theta=\tan^{-1}(y/x)$. The singular value of ψ near $r=0$ is a consequence of (4.12) not being valid near $x=0$. A sketch showing these perturbation streamlines is shown in figure 4. This shows that the physical roles of the external flow region are to provide the flow sucked in by the viscous wake and to enable the pressure gradient to drop to zero from its value in the viscous wake.

The most important point to note about the solution for this region is that it does not control the flow in the viscous wake, in this way being different from Stewartson's (1969) 'middle deck' which in his problem does influence the viscous 'lower deck'. Therefore the form of $U(y)$ in this region does not matter, although our particular solution for the external flow, (4.16), is only valid if $U(y)$ is such that $\beta l \ll \alpha$. For the solution of the viscous wake, (4.9), to be valid it is only necessary that there should be an uniform shear flow in the wake region, i.e.

$$\beta l R^{-\frac{1}{3}}\ll\alpha. \tag{4.17}$$

As a final comment on the solution (4.9), it is interesting to compare this exact solution with an approximate solution $\tilde{u}$ obtained by ignoring the term $(v^*-v^*_\infty)$ in (2.17). Then, of course, the wake does not satisfy (4.1), but (2.17) can be solved directly. On integrating twice as before, we find

$$\int_0^\infty y^{*2}\tilde{u}\,dy^*=I^*,$$

and then, in terms of I^*, $$u^* = \frac{I^*}{x^*}\frac{(\frac{1}{9}\eta)^{\frac{1}{3}}\, e^{-\frac{1}{9}\eta}}{3\Gamma(\frac{4}{3})}. \quad (4.18)$$

In figure 3 we have plotted $(\tilde{u}x^*/I^*)$ against $y/(x\nu/\alpha)^{\frac{1}{3}}$ to show that the maximum value of u is only about 13 % higher than u^*; also the shapes of the two curves are quite similar, although, as one might anticipate, the effect of omitting the $(v^* - v^*_\infty)$ term is to reduce the thickness of the wake.

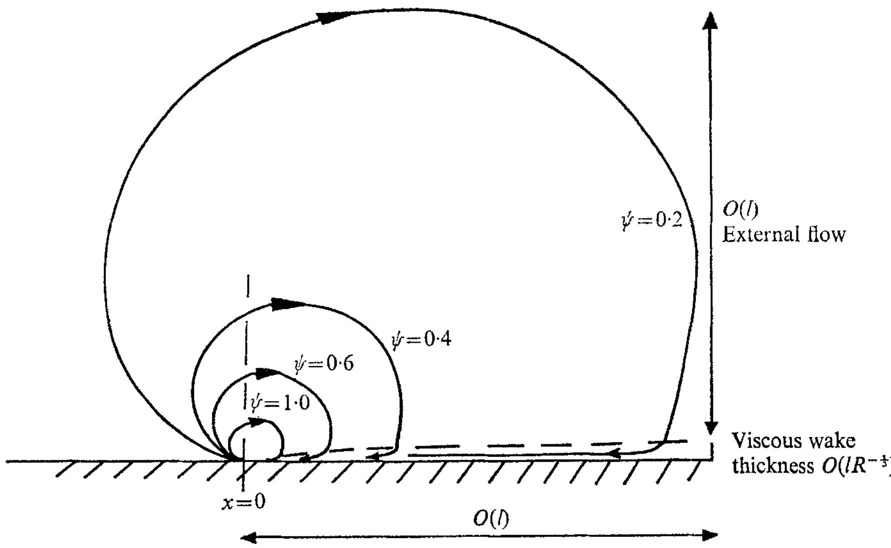

FIGURE 4. Perturbation streamlines of the external flow produced by the wake.

5. Complete solution for flow over a small hump†

Now consider the flow over a small hump of length $2b$ with the profile
$$y = bh^*(x^*), \quad \text{where} \quad x^* = x/b$$
and where $$h^* \ll 1. \quad (5.1)$$
Consequently we assume that flow over the hump does not separate and is a small perturbation on the boundary-layer flow, *everywhere* (see figure 5). We also assume that bh^*/δ is sufficiently small that in the viscous wake and external flow regions $U(y)$ is given adequately by
$$U = \alpha y. \quad (5.2)$$

To analyze the whole flow we assume $R_b = \alpha b^2/\nu \gg 1$ and we consider two regions as before. In the inner region we use a co-ordinate system curved round the hump:
$$\left.\begin{aligned} x^* &= x/b, \quad y^* = [y/b - h^*(x^*)]\,R_b^{\frac{1}{3}},\\ u_1 &= \alpha[y - bh^* + \epsilon b R_b^{-\frac{1}{3}}(\partial\psi^*/\partial y^*)_{x^*}],\\ u_2 &= \alpha[(y - bh^*)\,h^{*\prime} - \epsilon b R_b^{-\frac{2}{3}}(\partial\psi^*/\partial x^*)_{y^*} + \epsilon b R^{-\frac{1}{3}} h^{*\prime}\,\partial\psi^*/\partial y^*],\\ p &= P + \rho\epsilon(\alpha b R_b^{-\frac{1}{3}})^2 p^*, \end{aligned}\right\} \quad (5.3)$$

† I am indebted to Professor Stewartson for the idea of performing this analysis to check the previous results.

where ψ^* and p^* are non-dimensional functions of x^* and y^*. Then to $O(\epsilon)$, and ignoring terms $O(R_b^{-\frac{2}{3}})$, the equations of motion become

$$y^*\partial^2\psi^*/\partial x^*\partial y^* - \partial\psi^*/\partial x^* = -\partial p^*/\partial x^* + \partial^3\psi^*/\partial y^{*3}, \tag{5.4}$$

$$(\epsilon R^{\frac{1}{3}})^{-1} y^{*2} h^{*\prime\prime} = -\partial p^*/\partial y^*. \tag{5.5}$$

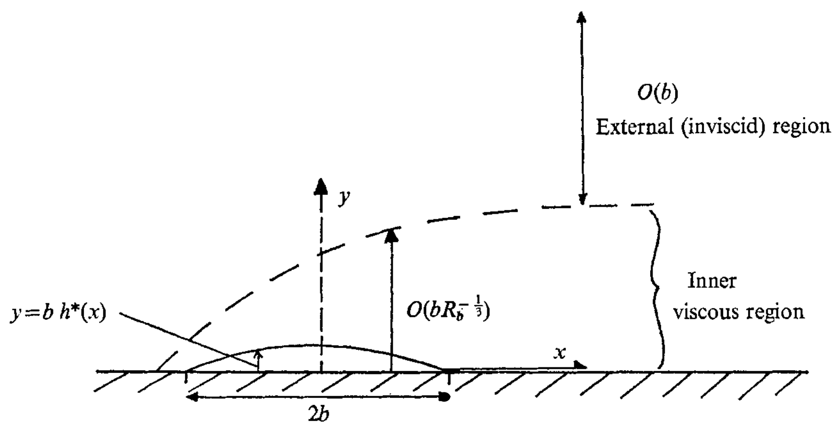

FIGURE 5. Flow over a small hump.

In the external inviscid region we use the same co-ordinate system as in §2, namely

$$\left.\begin{aligned} x^{**} &= x/b, \quad y^{**} = y/b, \\ u_1 &= \alpha(y + \epsilon b R_b^{-\frac{2}{3}}\, \partial\psi^{**}/\partial y^{**}), \\ u_2 &= -\epsilon\alpha b R_b^{-\frac{2}{3}}\, \partial\psi^{**}/\partial x^{**}, \\ p &= P + \rho\epsilon(\alpha l)^2 p^{**}, \end{aligned}\right\} \tag{5.6}$$

and then the governing equations, for the reasons mentioned in §4, become

$$\partial^2\psi^{**}/\partial x^{**2} + \partial^2\psi^{**}/\partial y^{**2} = 0.$$

The boundary conditions follows from the no slip condition, our hypothesis (H) and continuity of velocity at the boundary between the two regions. We find for the inner region

$$\left.\begin{aligned} &\left.\begin{aligned} y^* &= 0 \\ x^* &\to \infty \end{aligned}\right\} \quad \psi^* = \partial\psi^*/\partial y^* = 0. \\ &y^* \to \infty \quad \partial^2\psi^*/\partial y^{*2} \to 0, \quad \partial\psi^*/\partial y^* \sim (h^*/\epsilon) R_b^{\frac{1}{3}}, \\ \text{and} \quad &\psi^* \sim (h^*/\epsilon) R_b^{\frac{1}{3}} y^* + \psi^*_\infty(x); \end{aligned}\right\} \tag{5.7}$$

and for the outer region

$$\left.\begin{aligned} y^{**} &\to 0, \quad \psi^{**} = \psi^*_\infty(x), \\ (y^{**2} + x^{**2}) &\to \infty, \quad |\nabla\psi^{**}| \to 0. \end{aligned}\right\} \tag{5.8}$$

Here $\psi^*_\infty(x)$ is an unknown function to be found. It follows from (5.7) that $\epsilon = O[h^* R_b^{\frac{1}{3}}]$, and thence that if $\epsilon \ll 1$,

$$h^* \ll R_b^{-\frac{1}{3}}. \tag{5.9}$$

Physically this implies that the hump is buried inside the viscous region it creates, as shown in figure 5.

To solve (5.4), (5.5) subject to (5.7), first take the curl and then take the Fourier transform (F.T.) of ψ^*. We obtain

$$\partial^4\overline{\psi}/\partial y^{*4} - iky^* \,\partial^2\overline{\psi}/\partial y^{*2} = 0, \tag{5.10}$$

where $$\overline{\psi}(k, y^*) = \int_{-\infty}^{\infty} \psi^*(x^*, y^*)\, e^{-ikx^*}\, dx^*.$$

Putting $$\overline{X}(z) = \partial^2\overline{\psi}/\partial y^{*2},$$

where $z = y^* k^{\frac{1}{3}} e^{-\frac{1}{2}i\pi}$, (5.10) becomes

$$\partial^2\overline{X}/\partial z^2 - z\overline{X} = 0. \tag{5.11}$$

The boundary conditions on X follow from (5.7):

$$y \to \infty, \quad \overline{X} \to 0, \tag{5.12}$$

$$y = 0, \quad \partial\overline{X}/\partial z = -k^{\frac{2}{3}}\overline{\psi}_\infty(k), \tag{5.13}$$

and $$\int_0^\infty \overline{X}\, dz = k^{\frac{1}{3}} e^{-\frac{1}{2}i\pi} (\overline{h}/\epsilon)\, R_b^{\frac{1}{3}}, \tag{5.14}$$

where $\overline{h}$ and $\overline{\psi}_\infty$ are the F.T. of $h^*(x^*)$ and $\psi^*_\infty(x^*)$. The solution to (5.11) can be found in terms of Airy functions, the conditions (5.12) and (5.14) defining the solution as:

$$\overline{X} = G\overline{h}(k)\, \mathrm{Ai}\,(z\, e^{\frac{2}{3}\pi i}) \tag{5.15}$$

where $$G = (R_b^{\frac{1}{3}}/\epsilon)\, 3 e^{\pm\frac{1}{6}i\pi} |k|^{\frac{1}{3}} \quad \text{for} \quad k \gtrless 0.$$

Then, from (5.13)

$$\overline{\psi}(k) = 3(\overline{h}/\epsilon)\, R_b^{\frac{1}{3}} |k|^{-\frac{1}{3}} e^{\mp\frac{1}{6}i\pi} \mathrm{Ai}'\,(0). \tag{5.16}$$

It is not our intention here to provide a complete description of the flow over the hump, but rather to show that the exact solution (5.15) is equivalent to the similarity solution (4.9) as $x^* \to \infty$, and that the relation (3.10) in this case is exact, i.e. the constant in the similarity solution exactly satisfies

$$I = -C_1/\rho,$$

where C_1 is the component of the couple on the hump defined by (3.6). The analysis is lengthy but straightforward. Let

$$\overline{X}(k, y^*) = \overline{X}_1(k, y^*)\, \overline{h}(k), \tag{5.17}$$

then, since $h(x^*) = 0$ for $|x^*| > 1$, by the convolution theorem it follows that as $x^* \to \infty$,

$$X \sim X_1(x^*, y^*) \int_{-1}^{1} h^*(\zeta)\, d\zeta. \tag{5.18}$$

To find $\partial\psi^*/\partial y^*$ as $x^* \to \infty$ it is more convenient to find the inverse transform of $\overline{Y}(k, y^*)$ where

$$Y(x^*, y^*) = \int_{x^*}^{\infty} X\, dx^*.$$

Using (5.14) and (5.17) it it shown in the appendix that

$$Y(x^*, y^*) \sim -\frac{\left(\int_{-1}^{1} h^*(\zeta)\, d\zeta\right)(R_b^{\frac{1}{3}}/\epsilon)\,\eta^{\frac{1}{6}}\, e^{-\frac{1}{18}\eta}\, K_{\frac{1}{6}}(\frac{1}{18}\eta)}{(2\pi^{\frac{3}{2}}/3^{\frac{1}{2}})\,(x^*)^{\frac{1}{3}}} \tag{5.19}$$

where $$\eta = y^{*3}/x^* = y^3/[x\nu/\alpha].$$

Defining the perturbation velocity u as in §2, it follows from (5.3) that

$$u = \epsilon\alpha b\, R_b^{-\frac{1}{3}}\, \partial\psi^*/\partial y^*, \tag{5.20}$$

whence (5.19) gives the same expression for u as (4.9), where the constant

$$I = -\nu\alpha b^2 \int_{-1}^{1} h^*(\zeta)\, d\zeta. \tag{5.21}$$

As regards the external flow region, it is a simple matter to deduce from (5.15) that as $x^* \to \infty$

$$-\partial\psi^*/\partial x^* = v_\infty^* = I^* A x^{*-\frac{5}{3}},$$

in agreement with (4.12).

To calculate the component of the couple, C_1, exerted on the hump by the flow, we first consider the pressure on the body. Since $\partial p^*/\partial y^* = 0$ in the viscous region near the body, we only have to find $\partial p^*/\partial x^*$, which is of order $\partial\psi^*/\partial x^*$ as $y^* \to \infty$. Therefore from (5.7) and (5.15) we find

$$\partial p^*/\partial x^* = O[hR^{\frac{1}{3}}/\epsilon],$$

and thence the contribution to C_1 from pressure forces, namely

$$\int_0^{h_0} y(p_- - p_+)\, dy = O[\rho\alpha^2 b^4 h^{*3} R_b^{-\frac{1}{3}}].$$

On the other hand, the main contribution from the shear stress term is

$$\int_{-b}^{b} h\tau_{xy}\, dx = \rho\nu b^2\alpha \int_{-1}^{1} [1 + \epsilon b R_b^{-\frac{1}{3}}\, \partial^2\psi^*/\partial y^{*2}]\, h^*(x^*)\, dx^*$$
$$= \rho\alpha^2 b^4 R_b^{-1} \int_{-1}^{1} h^*(x^*)\, dx^*.$$

But in (5.9) we have assumed that $h^* \ll R_b^{-\frac{1}{3}}$ and therefore the couple produced by the pressure term is negligible compared to that by the shear stress. Thence

$$C_1 = \rho\nu l^2\alpha \int_{-1}^{1} h^*(\zeta)\, d\zeta.$$

Thus, from (5.21), $$C_1 = -I/\rho, \quad \text{Q.E.D.} \tag{5.22}$$

Note that for this hump $C_2 = O(\rho\alpha^2 b^4 R_b^{-\frac{1}{3}} h^*)$, so that the major contribution to the couple C comes from C_2 rather than C_1. Therefore our result (5.22) is only significant in its demonstration that (3.10) is exact in at least one case.

6. On the assumption of the analysis

The detailed assumptions which were needed to obtain the similarity solution (4.9) for the viscous wake were the following:

$$h/\delta \ll 1, \tag{6.1}$$

$$R = \alpha l^2/\nu \gg 1, \tag{6.2}$$

and when

$$x \sim l, \quad \epsilon \ll 1. \tag{6.3}$$

Given (6.1) we assumed that the following were also true

$$\lambda = l/L \ll 1, \tag{6.4}$$

and

$$\mu = (\beta/\alpha)\, lR^{-\frac{1}{3}} \ll 1. \tag{6.5}$$

First consider the implications of the assumptions (6.2) and (6.3). If U_1 is the free-stream velocity and L the distance in which the boundary layer changes,

$$\alpha \sim (U_1/L)\, R_L,$$

where $R_L = U_1 L/\nu$, so that (6.2) implies

$$l/L \gg R_L^{-\frac{3}{4}}. \tag{6.6}$$

Thus (6.6) and (6.4) provide bounds on l/L. In considering (6.3) we have to fix $R_h = \alpha h^2/\nu$, the Reynolds number of the body, and let us assume for convenience that $R_h \sim 1$ or greater. (We shall use $\sim$ hereafter to denote 'is of the order of magnitude of' in a 'factor of ten' sense.) Then if the body is bluff, $C_1 \sim \rho\alpha^2 h^4$ and assuming that (3.10) gives the right order of magnitude for I in terms of C_1, from (4.9) it follows that in the wake, if

$$\epsilon_x = -u/U,$$

$$\epsilon_x = -u/\alpha(x\nu/\alpha)^{\frac{1}{3}} \sim R_h^{\frac{4}{3}}(x/h)^{-\frac{4}{3}}. \tag{6.7}$$

Thus we can now make the inequality (6.3) more precise. For if ϵ is the largest value of ϵ_x that is tolerable for accuracy of the solution, (say 0·1),

$$l/h \sim (x/h) \geqslant \epsilon^{-\frac{3}{4}} R_h. \tag{6.8}$$

Since $\epsilon \ll 1$, it follows that (6.8) is consistent with (6.2). For the rather different problem of the small hump, the similarity solution is valid when

$$x/b \gg 1,$$

if $R_b \sim 1$ or greater. Thus for a blunt body the similarity solution is only valid a large number $O(R_h)$ body heights downstream, but for a small hump it is valid a few body lengths downstream.

Condition (6.4) is a simple physical way of expressing the condition that in the wake

$$\lambda_x = \left(\frac{d\alpha/dx}{\alpha}\right)\left(\frac{u^*}{\partial u^*/\partial x}\right) \ll 1. \tag{6.9}$$

Since $(d\alpha/dx)/\alpha$ is constant along the wake but $u^*/(\partial u^*/\partial x)$ increases as x^*, (6.9) cannot be true sufficiently far downstream. Therefore we assume that $\lambda_x < \lambda \ll 1$, λ being the highest value that is tolerable for the accuracy of the solution, (say, 0·1). Now for a laminar boundary layer $L \sim \delta R_\delta$ where $R_\delta = \alpha\delta^2/\nu$, and therefore, for whatever type of body, to satisfy (6.4)

$$x/h \lesssim \lambda(\delta/h)^3 R_h. \tag{6.10}$$

Thus given (h/δ), (6.10) gives the upper limit on (x/h) for the validity of the solution. Taking (6.10) and (6.8) together we obtain the following imprecise but necessary conditions on (h/δ) for a blunt body:

$$(h/\delta) \ll \epsilon^{\frac{1}{4}}, \quad (h/\delta) \ll \lambda^{\frac{1}{3}}. \tag{6.11}$$

From (6.11), (6.8), and (6.6) we can obtain useful criteria for R_h, R_L and ϵ, namely

$$\epsilon^{\frac{1}{2}} R_L \gg R_h \gg \epsilon^{\frac{1}{2}} R_L^{\frac{1}{2}}, \tag{6.12}$$

where R_h is assumed to be $O(1)$ or greater. Provided (6.10) is satisfied there is a range of (x/h) such that the criteria (6.3) and (6.4) are satisfied.

To satisfy the criterion (6.5) we must take l to be the largest value of x consistent with (6.9). Therefore

$$\mu \sim (\beta\delta/\alpha)\lambda^{\frac{1}{3}}.$$

Since $\beta\delta/\alpha \sim 1$, and, by definition $\lambda \ll 1$, (6.5) is satisfied. (If we consider higher-order terms in the expansion of $U(y)$, condition (6.9) still ensures (6.5) is valid.)

Thus we conclude that, if to the original assumptions (6.1) to (6.3) are added (for a blunt body) the assumptions in (6.10), then the similarity solution (4.9) is valid and the special assumptions (6.4) and (6.5) are unnecessary. In deriving (6.10) we have assumed $R_h \sim 1$ or greater. This assumption is not necessary because, as we have already stated, the solution (4.9) is independent of R_h. However, if $R_h \ll 1$, then new criteria for (6.7) and (6.10) must be found, which is a simple matter to do.

7. Conclusions

In this paper we have first of all shown that the wake far downstream of a body placed at the bottom of a boundary layer has properties which are quite different from those of a wake in a uniform flow. The form of the profile of the perturbation velocity, u, in the wake shown in figure 3 is different in the way one might expect, because $u = 0$ at $y = 0$. A more important difference between the velocity in these wakes and in those behind bodies in a uniform flow is that here the perturbation velocity decreases with the distance downstream in proportion to x^{-1} as opposed to $x^{-\frac{1}{2}}$ in the latter case, i.e. much faster. The other difference concerns the second-order flow outside the wake. Since in this case the flux of u^*, $\int_0^\infty u^* dy^*$, is not constant along the wake, the wake acts as a sink along its length, as opposed to providing a sink at infinity (Prandtl & Tietjens 1934, §80). In both cases there is a line source at the body, although in our case the source strength is singular. The external flow produced by this line sink is quite different, for example the velocity decays as $r^{-\frac{5}{3}}$ with distance r from the body, as opposed to r^{-1} in the other case. The last difference is that these kinds of wakes create a pressure gradient $\partial\tilde{p}/\partial x$ caused by the shear of the boundary layer and the sink flow. An important role of the external flow region is to reduce this pressure gradient to zero as $y^{**} \to \infty$.

The second point we have made is that the velocity in the wake is related to the couple on the body. There is little purpose in calculating the velocity in the wake if we cannot relate it to the size of the body and the forces on it. For a body in a uniform flow there is an exact relation between an integral of u which is constant along the wake, in this case the *momentum deficit*, and the drag on the body. So accurate is it that the drag can be calculated from measurements in the wake. However in our case the constant integral of u in the wake,

$$I = \int_0^\infty (\tfrac{3}{2})\,\alpha y^2\,dy,$$

is only approximately related to a component of the couple on the body C_1, defined by (3.6). This is not very satisfactory but it does mean that u can at least be estimated to an order of magnitude, which without this relationship would not be possible. Clearly this is a problem worthy of further study.

We began by mentioning the problem of transition. It would now be interesting to see whether a stability analysis of the velocity profiles in the wake would predict, even approximately, the correct Reynolds number of transition. It is immediately clear from the Orr–Sommerfeld equation that, since a uniform shear flow is stable, small disturbances can only grow if

$$\epsilon_x = -u(x,y)/\alpha y > 0.$$

Since we have shown in (6.6) that

$$\epsilon_x \sim R_h^{\frac{4}{3}}(x/h)^{-\frac{4}{3}},$$

ϵ_x decreases as x increases. However, the Reynolds number of the wake at the section where these disturbances grow, R_w, must be greater than some critical number $R_{\mathrm{crit}}(\epsilon_x)$, found from the Orr–Sommerfeld equation. Since

$$R_w = \alpha(x\nu/\alpha)^{\frac{2}{3}}/\alpha \sim x^{\frac{2}{3}}(\alpha/\nu)^{\frac{1}{3}},$$

R_w increases as x increases. Therefore the position at which the disturbances first grow lies at some large but finite value of (x/h) along the wake, as observed for various types of body by Fage (1943). Further qualitative results can be deduced assuming a plausible form for $R_{\mathrm{crit}}(\epsilon_x)$, but these will be more convincing when the stability analysis is completed.

Unfortunately the experimental results which might quantitatively be compared with the theory are not in a suitable form, and the experimental measurements made are not detailed enough. First, Liepmann & Fila (1947) disclaimed any quantitative accuracy, and then Tani & Sato's (1956) and Hall's (1968) results cannot be used because of their small-scale graphs and the small number of velocity profiles. However, the experiments confirm our main hypothesis (H) that the flow reverts to its upstream value if the body is small enough.

Also Tani & Sato's results confirm the general form of the velocity profile in the wake downstream of the bubble. In one case the wake extends over a distance $70 < x/h < 200$, where $R_h \simeq 150$ and $h/\delta = 0{\cdot}15$. According to our theory, with $\epsilon = \lambda = 0{\cdot}1$ and inserting appropriate constants into the order of magnitude expressions (6.7) and (6.10) the criteria (6.8) and (6.10) imply that $40 < x/h < 400$, and the criterion (6.11) implies that $h/\delta \ll 0{\cdot}46$. Thus experimental parameters *can* fall within the range where the theory is applicable. Clearly more experiments are needed in which detailed comparisons with the theory are possible.

This work was begun at the Central Electricity Research Laboratories at Leatherhead, and I am grateful for the advice of R. A. Scriven and J. Armitt. The particular form of the solution in §4 is due to Scriven, being more elegant than my clumsy version. I am also indebted to Dr H. K. Moffatt and the referees.

Appendix

From (5.15) and (5.17)

$$X_1(x^*, y^*) = \frac{3R_b^{\frac{1}{3}}/\epsilon}{2\pi}\int_{-\infty}^{\infty} e^{\pm\frac{1}{6}i\pi}|k|^{\frac{1}{3}}\,\mathrm{Ai}\,(y^*|k|^{\frac{1}{3}}\,e^{\pm\frac{1}{6}i\pi})\,e^{ikx^*}\,dk. \tag{A 1}$$

But

$$\mathrm{Ai}(z) = z^{\frac{1}{2}}\,K_{\frac{1}{3}}(\zeta)/(\pi 3^{\frac{1}{2}})$$

where $\zeta = \frac{2}{3}z^{\frac{3}{2}}$, and if

$$Y_1(x^*, y^*) = \int_{x^*}^{\infty} X_1(\zeta)\,d\zeta,$$

we find

$$Y_1(x^*, y^*) = -\frac{3(R_b^{\frac{1}{3}}/\epsilon)\,y^{*\frac{1}{2}}}{2\,(3^{\frac{1}{2}})\,\pi^2}\int_{-\infty}^{\infty} e^{\mp\frac{1}{4}i\pi}|k|^{-\frac{1}{2}}\,K_{\frac{1}{3}}(\lambda\,e^{\pm\frac{1}{4}i\pi}|k|^{\frac{1}{2}})\,e^{ikx^*}\,dk, \tag{A 2}$$

where $\lambda = \frac{2}{3}(y^*)^{\frac{3}{2}}$. Thence using the convolution theorem

$$Y_1(x^*, y^*) = \frac{3(R_b^{\frac{1}{3}}/\epsilon)\,y^{*\frac{1}{2}}}{\pi 3^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(x^*-\xi)\,g(\xi)\,d\xi, \tag{A 3}$$

where

$$f(x^*) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mp\frac{1}{3}i\pi}|k|^{-\frac{2}{3}}\,e^{ikx^*}\,dk,$$

and

$$g(x^*) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm\frac{1}{2}i\pi}\,|k|^{\frac{1}{6}}\,K_{\frac{1}{3}}\,(\lambda\,e^{\pm\frac{1}{4}i\pi}\,|k|^{\frac{1}{2}})\,e^{ikx^*}\,dk.$$

From the transforms of Erdélyi, Magnus & Oberhettinger (1954, pp. 10, 68, 55), we find

$$\left.\begin{aligned} f(x^*) &= \frac{\Gamma(\frac{1}{3})}{\pi}\sin\,(\tfrac{2}{3}\pi)\,(x^*)^{-\frac{1}{3}} && (x^* > 0),\\ &= 0 && (x^* < 0),\\ g(x^*) &= (\lambda^{\frac{1}{3}}/2^{\frac{4}{3}})\,x^{*-\frac{4}{3}}\,e^{-\lambda^2/(4x^*)} && (x^* > 0),\\ &= 0 && (x^* < 0).\end{aligned}\right\} \tag{A 4}$$

Substituting the results of (A 4) into (A 3), and making the transformation

$$1/\xi = (1/x^*) + t,$$

(A 3) becomes
$$Y_1(x^*, y^*) = -\frac{3(R_b^{\frac{1}{3}}/\epsilon)\, y^{*\frac{1}{2}}}{\pi\, 3^{\frac{1}{2}}\, \Gamma(\frac{2}{3})} \int_0^\infty \frac{e^{-\frac{1}{4}\lambda^2 t}\, dt}{t^{\frac{1}{3}}\,(t+1/x^*)}.$$

Then using the Laplace transform quoted by Erdélyi *et al.* (1954, p. 137)

$$Y_1(x^*, y^*) = -\frac{3^{\frac{1}{2}}(R_b^{\frac{1}{3}}/\epsilon)}{2\pi^{\frac{3}{2}}} \frac{\eta^{\frac{1}{6}}}{(x^*)^{\frac{1}{3}}} e^{-\frac{1}{18}\eta} K_{\frac{1}{6}}(\tfrac{1}{18}\eta),$$

the result which is quoted in (5.18).

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